Situation Dill01: Counting Arguments/Binomial Theorem

Prompt

During a class discussion, a student mentions that she has two brothers and one sister. Another student then asks, “What are the odds of that?”

Commentary

Students have a great curiosity about probability and are intrigued by answering questions involving it. Many students feel they know what the probability of this situation should be (many students feel the answer should be 50%), so the idea of counter-intuitive solutions can also be addressed. This prompt has multiple entry points for differing levels of students and has extensions that include constructing the binomial theorem. Additionally, this prompt can serve as a springboard for using multiple representations, such as the use of a bar graph and of a table of values. Counting the possible outcomes is one focus of this prompt, which requires systematic listing, tree diagram, and/or combinations/permutations.

Mathematical Foci

Mathematical Focus 1

The theoretical probability and odds are ratios that state the likelihood of an event occurring.

Odds are a ratio that expresses the probability of an event as the number of desired outcomes to the number of non-desired outcomes. So, for the roll of a six-sided die (or number cube), the odds of rolling a three are 1:5 because there is one desired outcome and five non-desired outcomes. The theoretical probability is almost the same, as it is the ratio of desired outcomes to total outcomes, or 1:6 in this case. There is much leeway in the use of the term odds. Students are familiar with odds from the news as a way of assigning payouts. The difference between payout odds and the definition used here should be made clear. The odds 10:1 (in a payout) mean that a one dollar bet receives ten dollars back: that is, ten times the original bet. A similar definition of odds refers to the ratio of the probability of an event occurring to the probability of an event not occurring. However, many popular uses equate odds with the theoretical probability.

Mathematical Focus 2

Experimental probability can be calculated with simulation.

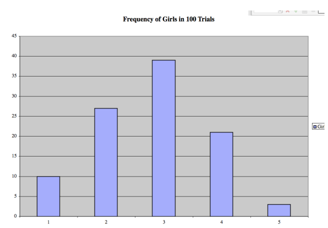
Experimental probability is the ratio of desired outcomes to total outcomes produced as a result of repeated trials. The first step students must decide is whether the trials are independent or dependent. In the prompt, the outcomes are generally considered to be independent (having no effect on one another). Assumptions about the outcomes are also made. For the prompt, it is usually assumed the likelihood of a male or a female is the same. This leads to assigning numbers to each outcome. Female may be one, while male is zero, or female may be one to fifty, while male is fifty-one to one hundred. An important part of this focus is that simulations look at outcomes in the long run. A set of

four or five trials is not appropriate for approximating the probability requested in this prompt. Optimally, several hundred (or more) trials need to be run. Comparing experimental probabilities helps establish the need for a large number of trials. Comparing results also shows that in the long run, the experimental probabilities converge toward the same value.
Running the simulation requires a random number generator. Use of a table, an electronic number generator, on-line resources, coin tossing or other physical devices gives access to different levels of students.

Mathematical Focus 3

Analysis of outcomes is best accomplished with multiple representations.
The results of a simulation should be analyzed in different formats. Formats to use include tables and bar graphs

Girls	Freq
0	10
1	27
2	39
3	21
4	3



Determining the set of all possible outcomes may use systematic list making, tree diagrams, or acting the situation out.

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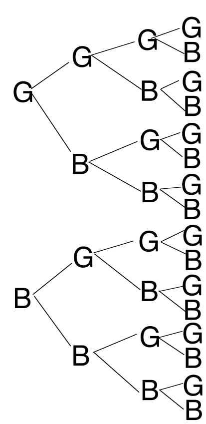


List making includes the concept of a set of all girls or all boys, then replacing one girl with a boy and looking for the different orders that can occur. This connects to the next focus. Additionally, counting the number of outcomes per child and multiplying the counts together ($2 \times 2 \times 2 \times 2$) counts the total number of outcomes.

Mathematical Focus 4

Permutation and combination are two ways to count outcomes.
Finding the different orders of one boy and three girls is an example of distinguishable permutations. Three of the children are girls and one a boy, which, by the fundamental counting principle, should have 4! = 24 different orders. But in this prompt, the gender "girl" is the same for three of the outcomes. Through investigation of leading questions such as, "How many orders of the letters in exile are there? How many orders of the letters in tomato are there? How many of them are different?" a general formula of $\frac{n!}{m_1!m_2!...m_k!}$ (*n* is the number of letters in the word, m_i is the number of items in each set of repeated letters) is discovered.
When the order of items matters, it is called a permutation. Counting the number of permutations can be done by listing or other methods, but eventually, there is a need for a formula to make the counting quicker. The pattern for permutations is discovered in a process like that for counting distinguishable permutations. The permutations of *n* things chosen *n* at a time is merely *n!*, but the permutation of *n* things chosen *k* at a time (notation ${}_nP_k$) requires looking at patterns to create a formula of $\frac{n!}{(n-k)!}$ wherein the *n!* is the number of permutations, but the letters left out, *n-k*, have their permutations divided away. Combinations are counting the number of ways *n* things can be chosen *k* at a time. Starting with the permutation formula, a similar formula is found for counting combinations. Of the $\frac{n!}{(n-k)!}$ permutations, *k* items are used. These *k* items have *k!* orders, once again using the fundamental counting principle. Since orders do not matter with combinations, the *k!* orders are divided from the number of permutations, giving $\frac{n!}{k!(n-k)!}$. The two most common notations are ${}_nC_k$ and $\left(\begin{array}{c} n \\ k \end{array}\right)$.
The three counting methods fit together for this prompt. An easy way to attack the problem of the number of ways two boys and two girls may be born is to think of finding the number of combinations of two boys are two girls in four children. Using $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$ gives the desired result of six. However, since order does not really count for the genders, this is actually an oversimplification. In reality, the counting is of distinguishable permutations of the four children, with two sets of girls and two sets of boys, that is $\frac{4!}{2!2!}$, which is the same formula as $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$, but a subtly different meaning.

Mathematical Focus 5

Pascal's Triangle may be used to count the outcomes.
By looking at the patterns that occur with one child, two children, three children, etc., and the possible gender outcomes, Pascal's Triangle can be constructed. Connections between choose notation/combinations, distinguishable permutations for a set with

exactly two elements, and the layout of the Triangle establish the pattern of Pascal's Triangle.

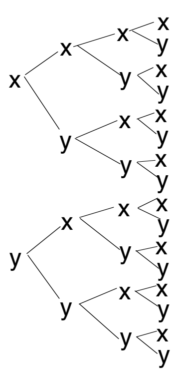
Row					
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Row 4 gives the results of the four children prompt. The total of each row being a power of two, in this case $2^4 = 16$, is also an important focus of this problem as it counts the total number of outcomes.

Mathematical Focus 6

The binomial theorem may be constructed using distinguishable permutations from counting outcomes.

The binomial theorem can be constructed using this prompt.
Converting the variable names to x and y shows:



or

xxxx = x^4
xxxy = x^3y
xxyx = x^3y

xxyy = x^2y^2
xyxx = x^3y
xyxy = x^2y^2
xyyx = x^2y^2
xyyy = xy^3
yxxx = xy^3
yxxy = x^2y^2
yxyx = x^2y^2

yxyy = xy^3
yyxx = x^2y^2
yyxy = xy^3
yyyx = xy^3
yyyy = y^4

Simplify $(x + y)^4$.

Consider the number of ways the variables *x* and *y* can be arranged.

$(x + y)^4 = 1x^4$

+ 1xxxy + 1xxyx + 1xyxx + 1yxxx

+ 1xxyy + 1xyxy + 1xyyx + 1yxxy + 1yxyx + 1yyxx

+ 1xyyy + 1yxyy + 1yyxy + 1yyyx
+ $1y^4$

This is showing the distinguishable permutations of the letters and is the same as the result $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
Using the notation from the earlier focus, this is

$$\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Further investigations include the relationship of the exponents and the numbers in the combinations notation, using powers other than one on the binomial that is used as a prompt and using different coefficients other than one. For example, this prompt may be rewritten as $\left(\frac{1}{2}g + \frac{1}{2}b\right)^4 = \frac{1}{16}g^4 + \frac{1}{4}g^3b + \frac{3}{8}g^2b^2 + \frac{1}{4}gb^3 + \frac{1}{16}b^4$, which gives the outcomes as girls and boys and the associated probability as the coefficient.

Post-commentary

There are many patterns in Pascal's Triangle that may be explored. Using probability as a basis makes expanding polynomials using the binomial theorem more immediately accessible as having an application, though the results are actually more easily found with other methods. The connection between the binomial theorem and the binomial distribution is a natural one that may follow easily.